Non-Monotone Perturbations for Nonlinear Parabolic Equations Associated with Subdifferential Operators (Nonlinear Functional Analysis)

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数理解析研究所講究録
428巻 1981年 11-33

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§ 1. Introduction.

In the present paper, we consider the existence and regularity of strong solutions of several types of problems for the following abstract equation (E) in a real separable Hilbert space H :

$$\text{(E)} \quad \frac{du}{dt}(t) + \partial\varphi^t(u(t)) + B(t,u(t)) \ni f(t) \ ,$$

where $f(t)$ is a given function, $\partial\varphi^t$ is the subdifferential of a time-dependent lower semi-continuous convex function φ^t from H into $[0,+\infty]$ with $\varphi^t \not\equiv +\infty$, and where $B(t,\cdot)$ is a possibly non-monotone multi-valued nonlinear operator with $D(B(t,\cdot)) \supset D(\partial\varphi^t)$, which is regarded as a perturbation for $\partial\varphi^t$ in a sense. Here and henceforth we are concerned with strong solutions of (E) in the following sense.

DEFINITION 1.1. A function $u(t)$ is said to be a strong solution of (E) in an open interval I of R^1, if the following properties (i) and (ii) are satisfied.

(i) $u(t)$ is an H-valued absolutely continuous function on any compact subset of I.

(ii) $u(t) \in D(\partial\varphi^t)$ for a.e. $t \in I$ and there exist two functions $g(t), b(t) \in L^2_{loc}(I;H)$ such that $g(t) \in \partial\varphi^t(u(t))$, $b(t) \in B(t,u(t))$

and $\quad du(t)/dt + g(t) + b(t) = f(t)\quad$ hold for a.e. $t \in I$.

Actually our main concern here is to study the existence of strong solutions for the following three types of problems :

(I) <u>Cauchy Problems $(E)_0$</u>: For each initial data a given in $\mathcal{B}_{\alpha,p}(\partial\varphi^0)$ ($0 < \alpha < 1/2$, $1 \le p \le +\infty$), interpolation classes between $\overline{D(\partial\varphi^0)}$ and $D(\partial\varphi^0)$, find a strong solution $u(t)$ of (E) in $(0,+\infty)$ with $u(+0) = a$.

(II) <u>Periodic Problems $(E)_\pi$</u> : When $\varphi^0(\cdot) = \varphi^T(\cdot)$, find a strong solution $u(t)$ of (E) in $(0,T)$ with $u(0) = u(T)$.

(III) <u>Almost-Periodic Problems $(E)_{\alpha\pi}$</u> : When $f(t)$ is an H-valued almost-periodic function and φ^t varies almost-periodically with respect to t in a sense, find an H-valued almost-periodic strong solution of (E) in R^1.

When $B(t,\cdot)$ is a monotone-type operator (or $B(t,\cdot) \equiv 0$), many results on the existence, uniqueness and regularity of strong solutions for $(E)_0$ and $(E)_\pi$ have been developed so far. In particular, we here refer to Brézis [9], Watanabe [30], Maruo [21], Attouch-Damlamian [2], Kenmochi [17], Yamada [31] and Yotsutani [33] for $(E)_0$, and Bénilan-Brézis [4], Nagai [23], Yamada [32] for $(E)_\pi$. On the other hand, when $B(t,\cdot)$ is not monotone, the study for $(E)_0$ has been made recently by several authors under some compactness assumptions on $D(\varphi^t) = \{u \in H ;\ \varphi^t(u) < +\infty\}$ similar to each other. For example, Attouch-Damlamian [3] and Biroli [5] dealt with the case where $\partial\varphi^t \equiv \partial\varphi$ and $B(t,\cdot)$ belongs to a class of time-dependent upper semi-continuous operators. The case that $\partial\varphi^t \equiv \partial\varphi$ and

$B(t,\cdot) = -\partial\psi$ was studied by Koi-Watanabe [18], Ishii [16] and the author [24,25]. Our main distinction here is not only to treat periodic problem $(E)_\pi$ but also to allow t-dependence of $D(\varphi^t)$ and to treat a much wider class of perturbations $B(t,\cdot)$ aiming at an applications to Navier-Stokes-type equations.

As for $(E)_{\alpha\pi}$, the study in this direction seems to be very few. When the perturbing term $B(t,\cdot)$ is absent, Biroli [6] studied the case $\partial\varphi^t \equiv \partial\varphi$. In relation to this problem, we also refer to Amerio-Prouse [1] and Foias [11], where the almost-periodic problem for the Navier-Stokes equations in cylindrical domains is teated. We shall discuss abstract problems $(E)_0$, $(E)_\pi$, $(E)_{\alpha\pi}$ and their applications for the Navier-Stokes equations in bounded regions with moving boundaries in §2, §3 and §4 respectively.

§ 2. Cauchy Problems.

2.1. Subdifferential operators and interpolation classes.

Let H be a real Hilbert space with the inner product $(\cdot,\cdot)_H$ and the norm $|\cdot|_H$, which are often denoted by $(\cdot,\cdot)$ and $|\cdot|$ respectively. We denote by $\Phi(H)$ the family of all lower semi-continuous convex functions φ from H into $(-\infty,+\infty]$ with $\varphi \not\equiv +\infty$. For each $\varphi\in\Phi(H)$, the effective domain $D(\varphi)$ of is defined by $D(\varphi) = \{u\in H\ ;\ \varphi(u) < +\infty\}$ and the sub-differential $\partial\varphi$ of φ is defined by

$$\partial\varphi(u) = \{f\in H\ ;\ \varphi(v) - \varphi(u) \geq (f, v-u)_H \quad \text{for all} \quad u\in H\}$$

with domain $D(\partial\varphi) = \{u\in H\ ;\ \partial\varphi(u) \neq \emptyset\}$.

Then, as is well known, $\partial\varphi$ is a maximal monotone in H, and $\overline{D(\varphi)}$, the closure of $D(\varphi)$ in the H-norm, coincides with $\overline{D(\partial\varphi)}$ (see Brézis [9]).

Let A be a maximal monotone operator in H with domain D(A), and put $J_\lambda = (I + \lambda A)^{-1}$, $\lambda > 0$. For each $\alpha \in (0,1)$ and $p \in [1,+\infty]$, intermediate classes $\mathcal{B}_{\alpha,p}(A)$ between $\overline{D(A)}$ and D(A) are defined by

$$\mathcal{B}_{\alpha,p}(A) = \{u \in \overline{D(A)} \; ; \; t^{-\alpha}|u - J_t|_H \in L^p_*(0,1)\},$$

where $L^p_*(0,1) = \{f \; ; \; |f|_{L^p_*} = (\int_0^1 |f(t)|^p \frac{dt}{t})^{1/p} < +\infty\}$, $1 \leq p < +\infty$, and $L^\infty_*(0,1) = L^\infty(0,1)$.

Then it is shown that $\mathcal{B}_{\alpha,p}(A) \subset \mathcal{B}_{\alpha,q}(A)$ for all $\alpha \in (0,1)$ if $1 \leq p < q \leq \infty$, and that $\mathcal{B}_{\alpha,p}(A) \subset \mathcal{B}_{\beta,q}(A)$ for all $1 \leq p,q \leq \infty$ if $0 < \beta < \alpha < 1$ (see D. Brézis [8]).

2.2. Local existence.

First of all we introduce the following three conditions, which will be assumed throughout this paper.

(A.φ^t) For each $t \in R^1$, $\varphi^t \in \Phi(H)$ and $\varphi^t \geq 0$. Furthermore, there exist constants $K \geq 0$, $\delta > 0$, $\beta \in [0,1]$, and a continuous monotone increasing function $m(\cdot)$ on $[0,+\infty)$ such that for each $t_o \in R^1$ and $x_o \in D(\varphi^{t_o})$, there exists a function $x(t)$ satisfying

(2.1) $\quad |x(t) - x_o|_H \leq m(|x_o|_H)\,|t - t_o|(\varphi^{t_o}(x_o) + K)^\beta$,

(2.2) $\quad \varphi^t(x(t)) \leq \varphi^{t_o}(x_o) + m(|x_o|_H)\,|t - t_o|(\varphi^{t_o}(x_o) + K)$,

for all $t \in [t_o - \delta, t_o + \delta]$.

(A.1) For each $t \in R^1$ and $L \in (0,+\infty)$, the set $\{u \in H \; ; \; \varphi^t(u) + |u|_H \leq L\}$ is compact in H.

In what follows, we always assume that $B(t,\cdot)$ is single valued, for the sake of simplicity.

(A.2) For each interval $[a,b]$ in R^1, the following (i) and (ii) are satisfied.

(i) $B(t,\cdot)$ is <u>measurable</u> in the following sense : If $u(t) \in C([a,b];H)$, $du(t)/dt \in L^2(a,b;H)$ and there exists a function $g(t) \in L^2(a,b;H)$ with $g(t) \in \partial\varphi^t(u(t))$ for a.e. $t \in [a,b]$, then $B(t,u(t))$ is measurable in $t \in [a,b]$.

(ii) $B(t,\cdot)$ is <u>demiclosed</u> in the following sense : If $u_n \to u$ in $C([a,b];H)$, $g_n \to g$ weakly in $L^2(a,b;H)$ with $g_n(t) \in \partial\varphi^t(u_n(t))$, $g(t) \in \partial\varphi^t(u(t))$ for a.e. $t \in [a,b]$, and if $B(t,u_n(t)) \to b(t)$ weakly in $L^2(a,b;H)$, then $b(t) = B(t,u(t))$ for a.e. $t \in [a,b]$.

Next, we introduce the following three types of boundedness conditions for $B(t,\cdot)$.

(A.3) There exist a function $M(\cdot) \in \mathcal{M}$ and a constant $k \in [0,1)$ such that

(2.3) $$|B(t,u)|_H^2 \leq k|\overset{\circ}{\partial\varphi}{}^t(u)|_H^2 + M(\varphi^t(u) + |u|_H) \quad \text{for all } t \in R^1 \text{ and } u \in D(\partial\varphi^t).$$

(A.4)$_\alpha$ For an exponent $\alpha \in (0,1/2)$, there exists a function $M(\cdot) \in \mathcal{M}$ such that

(2.4) $$|B(t,u)|_H \leq M(|u|_H)\{\varepsilon\,|\overset{\circ}{\partial\varphi}{}^t(u)|_H + M(\tfrac{1}{\varepsilon})\,|\varphi^t(u)|^{\frac{1-\alpha}{1-2\alpha}} + 1\}$$
$$\text{for all } \varepsilon > 0,\ t \in R^1 \text{ and } u \in D(\partial\varphi^t).$$

(A.5) There exist a constant $\gamma \in (0,1)$ and a function $M(\cdot) \in \mathcal{M}$ such that

(2.5) $$|B(t,u)|_H \leq M(|u|_H)\,(\,|\overset{\circ}{\partial\varphi}{}^t(u)|_H^{1-\gamma} + |\varphi^t(u)|^{1-\gamma} + 1\,)$$
$$\text{for all } t \in R^1 \text{ and } u \in D(\partial\varphi^t).$$

Here and henceforth, $\mathcal{M}$ denotes the family of all positive monotone increasing functions on $[0,+\infty)$, and $\overset{\circ}{\partial\varphi}{}^t$ the minimal section of $\partial\varphi^t$,i.e., $\overset{\circ}{\partial\varphi}{}^t(u)$ is the unique element of least norm in $\partial\varphi^t(u)$.

Then our local existence results are stated as follows according as initial data belong to $D(\varphi^0)$, $\mathcal{B}_{\alpha,p}(\partial\varphi^0)$ ($0<\alpha<1/2$), and $\overline{D(\varphi^0)}$.

THEOREM I Let $(A.\varphi^t)$,(A.1),(A.2) and (A.3) be satisfied. Let $a\in D(\varphi^0)$ and $f(t)\in L^2_{loc}([0,+\infty);H)$. Then there exists a positive number T depending on $|a|_H$ and $\varphi^0(a)$ such that $(E)_0$ has a strong solution $u(t)$ in $(0,T)$ satisfying

(2.6) $du(t)/dt$, $g(t)$, $B(t,u(t))\in L^2(0,T;H)$,

(2.7) $\varphi^t(u(t))$ is absolutely continuous on $[0,T]$.

THEOREM II Let $(A.\varphi^t)$,(A.1),(A.2) and $(A.4)_\alpha$ be satisfied. Let $a\in\mathcal{B}_{\alpha,p}(\partial\varphi^0)$ with $p\in[1,2]$ and $f(t)\in L^2_{loc}([0,+\infty);H)$. Then there exists a positive number T depending on $|a|_H$ and $|a|_{\alpha,p}$ (*1) such that $(E)_0$ has a strong solution $u(t)$ in $(0,T)$ satisfying

(2.8) $t^{1/2-\alpha}du(t)/dt$, $t^{1/2-\alpha}g(t)$, $t^{1/2-\alpha}B(t,u(t))\in L^2(0,T;H)$,

(2.9) $t^{-\alpha}|u(t)-a|_H$, $t^{1/2-\alpha}|\varphi^t(u(t))|\in L^q_*(0,T)$ for all $q\in[2,\infty]$.

THEOREM III Let $(A.\varphi^t)$,(A.1),(A.2) and (A.5) be satisfied. Let $a\in\overline{D(\varphi^0)}$ and $f(t)\in L^2_{loc}([0,+\infty);H)$. Then there exists a positive number T depending on $|a|_H$ such that $(E)_0$ has a strong solution $u(t)$ in $(0,T)$ satisfying (2.8) with $\alpha=0$ and

(2.10) $\varphi^t(u(t))\in L^1(0,T)$, $t\,\varphi^t(u(t))\in L^\infty(0,T)$.

(*1) $|a|_{\alpha,p}=|t^{-\alpha}|a-J_t a|_H|_{L^p_*}$, $J_t=(I+t\partial\varphi^0)^{-1}$.

Sketch of proof. These theorems can be proved in much the same way, so we here give a sketch of the proof for Theorem II.

Let X_S^α be a Banach space with the norm

$$\|u\|_{\alpha,S} = \left(\int_0^S t^{1-2\alpha}|u(t)|_H^2\,dt\right)^{1/2}, \quad 0<\alpha<1/2 .$$

For each $h(t) \in X_S^\alpha$, let us consider the equation :

$$(E)_o^* \begin{cases} (2.11) \quad du_h(t) + \partial\varphi^t(u_h(t)) \ni -h(t) + f(t), \; 0<t<S, \\ (2.12) \quad u_h(0) = a . \end{cases}$$

Then, under assumption $(A.\varphi^t)$, it is known that $(E)_o^*$ has a unique strong solution $u_h(t)$ in $(0,S)$ (see [31] and [33]). Therefore, for each $a \in \mathcal{B}_{\alpha,p}(\partial\varphi^0)$, $S \in (0,+\infty)$ and $f(t) \in L^2(0,S;H)$, we can define an operator $\mathbb{E}_{a,f,S}$ from X_S^α into $C([0,S];H)$ by $\mathbb{E}_{a,f,S}(h) = u_h$. Furthermore we introduce another operator $\mathbb{B}_{a,f,S}$ by $\mathbb{B}_{a,f,S}(h)(t) = B(t,\mathbb{E}_{a,f,S}(h)(t)) = B(t,u_h(t))$. Then, making good use of the nonlinear interpolation theory introduced by D. Brézis [8] and energy estimates for $(E)_o^*$, under assumptions $(A.\varphi^t)$ and $(A.4)_\alpha$, we find that for an appropriate positive number R and a sufficiently small S, $\mathbb{B}_{a,f,S}$ maps the set $K_{S,R}^\alpha = \{u \in X_S^\alpha;\ \|u\|_{\alpha,S} \le R\}$ into itself. In order to obtain energy estimates for $(E)_o^*$, we much rely on the following proposition.

PROPOSITION 2.1. Let $(A.\varphi^t)$ be satisfied and $u(t)$ be a continuous function on $[a,b]$ such that the set $\mathcal{L} = \{t \in [a,b];$ $du(t)/dt$, $d\varphi^t(u(t))/dt$ exist and $u(t) \in D(\partial\varphi^t)\}$. Then

$$\left|\frac{d}{dt}\varphi^t(u(t)) - \left(g, \frac{du}{dt}(t)\right)_H\right|$$
$$\le m(|u(t)|_H)\,|g|_H\,(\varphi^t(u(t))+K)^\beta + m(|u(t)|_H)\,(\varphi^t(u(t))+K)$$

holds for all $t \in \mathcal{L}$ and $g \in \partial\varphi^t(u(t))$.

Furthermore, by using energy estimates and a compactness argument (Ascoli's theorem) , we deduce the following continuity of $\mathbb{E}_{a,f,S}$ and $\mathbb{B}_{a,f,S}$.

<u>LEMMA 2.2</u>. Let $(A.\varphi^t)$, (A.1), (A.2) and (A.4) be satisfied. If $h^n \to h$ weakly in X_S^α as $n \to +\infty$, then $\mathbb{E}_{a,f,S}(h^n) \to \mathbb{E}_{a,f,S}(h)$ in $C([0,S];H)$ and $\mathbb{B}_{a,f,S}(h^n) \to \mathbb{B}_{a,f,S}(h)$ weakly in X_S^α as $n \to +\infty$.

Thus, for a sufficiently small S , $\mathbb{B}_{a,f,S}$ is a weakly continuous mapping form the weakly compact convex set $K_{S,R}^\alpha$ into itself. Then , by Schauder's fixed-point theorem , there exists an elememt $b = \mathbb{B}_{a,f,S}(b)$, i.e., $\mathbb{E}_{a,f,S}(b) = u$ satisfies

$$\begin{cases} du(t)/dt + \partial\varphi^t(u(t)) + b(t) \ni f(t) & \text{for a.e. } t \in (0,S), \\ b(t) = B(t,u(t)) & \text{for a.e. } t \in (0,S), \\ u(0) = a . \end{cases}$$

That is to say , $u(t)$ is the desired local strong solution of $(E)_0$ in $(0,S)$.

As for the cases $a \in D(\varphi^0)$ and $a \in \overline{D(\varphi^0)}$, we can apply the same idea as above by replacing X_S^α by $L^2(0,S;H)$ and $X_S^0 = \{ u ; (\int_0^S |u(t)|_H^2\, t\, dt)^{1/2} + (\int_0^S |u(t)|_H^{(2+\gamma)/2} dt < +\infty \}$ (γ is the exponent appearing in (A.5)) respectively.

<u>REMARK 2.3</u>. When $B(t,\cdot)$ is a multi-valued operator , as a matter of course , $\mathbb{B}$ becomes a multi-valued mapping. In this case, however, instead of Schauder's theorem , we can rely on Fan's fixed-point theorem for upper semi-continuous multi-valued mappings (see [3], [5] and [10]).

2.3. Global existence.

Firstly we give a sufficient condition which guarantees that every local strong solutions can be continued globally to $(0,+\infty)$.

THEOREM IV Let $(A.\varphi^t)$, (A.1), (A.2) and the following (A.6) be satisfied.

(A.6) There exist constants $\alpha > 0$, $C \geq 0$, $k \in [0,1)$ and a function $M(\cdot) \in \mathcal{M}$ such that

(2.11) $(-g - B(t,u), u)_H + \alpha \varphi^t(u) \leq C(|u|_H^2 + 1)$

for all $t \in (0,+\infty)$, $u \in D(\partial\varphi^t)$ and $g \in \partial\varphi^t(u)$,

(2.12) $|B(t,u)|_H^2 \leq k|\partial^\circ\varphi^t(u)|_H^2 + M(|u|_H)(\varphi^t(u) + 1)^2$

for all $t \in (0,+\infty)$ and $u \in D(\partial\varphi^t)$.

Let $f(t) \in L^2_{loc}([0,+\infty); H)$ with $\|f\|_{2,\infty} = \sup_{t>0} \int_t^{t+1} |f(s)|_H^2 ds < +\infty$.

Then every local strong solution of $(E)_o$ can be continued globally to $(0,+\infty)$ as a strong solution of $(E)_o$.

Proof. Let $u(t)$ be a strong solution of $(E)_o$ in $(0,S)$. Then it is easy to see that (2.11) gives a priori bounds for $\max_{0\leq t\leq S} |u(t)|_H + \int_0^S \varphi^t(u(t))\, dt$. Hence, by virtue of Proposition 2.1 and Gronwall's inequality, multiplying (1.1) by $g(t) = - du(t)/dt - B(t,u(t)) + f(t) \in \partial\varphi^t(u(t))$, we can obtain a priori bounds for $\varphi^t(u(t))$. Then the assertion of the theorem follows from Theorem I.

When condition (A.6) is absent, it is known that there are some cases where if a and $f(t)$ satisfy certain conditions, then the corresponding local strong solution $u(t)$ of $(E)_o$ blows up in a finite time T_m, i.e., $|u(t)|_H \to +\infty$, $\varphi^t(u(t)) \to +\infty$, etc. as $t \to T_m$ (see, e.g., Fujita [12], Tsutsumi [29], Ishii [16] and the author [25]). In such cases, however, it is quite often possible to continue local solutions globally if their data a and $f(t)$ are sufficiently small. This is also the case with our situation. To illustrate this, we introduce the following condition.

(A.7) The following (i) and (ii) are satisfied.

(i) $\varphi^t(0) = 0$ for all $t \in R^1$,

(ii) There exist positive constants k , α , C_1, p and functions $\ell(\cdot) \in \mathcal{M}$, $\ell_1(\cdot)$, $\ell_2(\cdot) \in \mathcal{M}_0 = \{ \ell(\cdot) \in \mathcal{M} ; \ell(r) \to 0$ as $r \to 0\}$ such that

(2.13) $|B(t,u)|_H^2 \leq \{ k + \ell_1(\varphi^t(u))\} |\partial^\circ\varphi^t(u)|_H^2 + \ell(\varphi^t(u))$, $0 \leq k < 1$,
for all $t \in R^1$ and $u \in D(\partial\varphi^t)$,

(2.14) $(- g - B(t,u) , u)_H + \alpha\varphi^t(u) \leq \ell_2(\varphi^t(u)) \cdot \varphi^t(u)$
for all $t \in R^1$ and $u \in D(\partial\varphi^t)$,

(2.15) $C_1 \; |u|_H^p \leq \varphi^t(u)$, $1 < p < +\infty$, for all $u \in D(\partial\varphi^t)$.

Then we have the following stability result.

<u>LEMMA 2.4</u>. Let (A.φ^t).(A.1),(A.2) and (A.7) be satisfied. Then there exist positive number N and r_0 such that for every $r \in (0,r_0)$, if $|a|_H + \varphi^0(a) \leq r^{1/(p-1)}$ and $\|f\|_{2,\infty} \leq r$, then every strong solution $u(t)$ of $(E)_0$ in $(0,T)$ enjoies the a priori estimate $\max \{|u(t)|_H + \varphi^t(u(t)) ; 0 \leq t \leq T\} \leq N r^{1/(p-1)}$ independent of T.

From this lemma , the following global extension result is derived .

THEOREM V Let (A.7) and all assumptions in Theorem I (resp. II or III) be satisfied. Then there exists a (sufficiently small) positive number r such that if $|a|_H + \varphi^0(a) \le r$ (resp. $|a|_H + \|a\|_{\alpha,p} \le r$ or $|a|_H \le r$), $\underbrace{\quad}_{\text{and } \|f\|_{2,\infty} \le r^{p-1}}$ then $(E)_0$ has a global strong solution in $(0,+\infty)$, i,e., the assertion of Theorem I (resp. II or III) holds true with $T = +\infty$.

2.4. Application.

Let $Q = \bigcup_{t\in R} Q(t) \times \{t\}$ be a non-cylindrical domain in $R^n_x \times R^1_t$ which is smooth in (x,t) in the following sense.

(A.Q) For each $t \in R^1$, $Q(t)$ is a bounded domain in R^n_x of C^3-class, and there exists a C^3-diffeomorphism $\Psi : Q_0 := \overline{Q(0)} \times R^1 \to \overline{Q}$ with $\Psi(x,t) = (F(x,t) , t)$ (level preserving) satisfying

(i) $F(x,0) = x$ for all $x \in Q(0)$,

(ii) $\sup \{D^m F(x,t) \ ; \ m = 0,1,2,3, \ (x,t) \in Q_0 \ , \ D = \frac{\partial}{\partial x} , \frac{\partial}{\partial t} \} < +\infty$,

(iii) $\inf \{\det(\frac{\partial F^i}{\partial x_j}(x,t)) \ ; \ (x,t) \in Q_0 \} > 0$.

Let us now consider the following initial-boundary value problem for the Navier-Stokes equation in $Q_+ = \bigcup_{t>0} Q(t) \times \{t\}$:

$$(Pr.NS)_0 \begin{cases} (2.16) \quad \dfrac{\partial u}{\partial t} - \Delta u + (u \cdot \nabla) u = f - \nabla p_* & \text{in } Q_+ , \\ (2.17) \quad \operatorname{div} u = 0 & \text{in } Q_+ , \\ (2.18) \quad u = 0 \quad \text{on } \Gamma_+ = \bigcup_{t>0} \partial Q(t) \times \{t\}, & \\ (2.19) \quad u = a(x) & \text{in } Q(0) , \end{cases}$$

where the unknown $u(x,t)$ and given $f(x,t)$, $a(x)$ are real n-dimensional vector functions , while the unknown $p_*(x,t)$ is a real scalar function.

This kind of problem has been investigated by several authors : Fujita-Sauer [14], Bock [7], Inoue-Wakimoto [15] and Ôtani-Yamada [27]. These contributions differ in methods and results. Our advantage here , as well as in [27], is that regularity of solution with respect to time t near boundary can be given explicitly. To formulate our results, we shall use the notations:

$C_\sigma^\infty(\Omega) = \{u = (u^1, u^2, \cdots, u^3);\ u^i \in C_0^\infty(\Omega), i = 1, 2, \cdots, n,\ \mathrm{div}\, u = 0\}$,

$\mathbb{H}(\Omega) = (L^2(\Omega))^n = \{u = (u^1, u^2, \cdots, u^n);\ u^i \in L^2(\Omega), i = 1, 2, \cdots, n\}$,

$\mathbb{H}_\sigma(\Omega)$ = the completion of $C_\sigma^\infty(\Omega)$ in the $\mathbb{H}(\Omega)$-norm,

P_Ω = the orthogonal projection from $\mathbb{H}(\Omega)$ onto $\mathbb{H}_\sigma(\Omega)$,

$\mathbb{H}_0^1(\Omega) = (H_0^1(\Omega))^n$, $\mathbb{H}^2(\Omega) = (H^2(\Omega))^n$, $\mathbb{H}_\sigma^1(\Omega) = \mathbb{H}_0^1(\Omega) \cap \mathbb{H}_\sigma(\Omega)$,

A_Ω = the Stokes operator $-P_\Omega\Delta$ with domain $D(A_\Omega)$
$= \mathbb{H}^2(\Omega) \cap \mathbb{H}_\sigma^1(\Omega)$,

A_Ω^α = the fractional power of A_Ω of order $\alpha > 0$.

Results. (1) The case $n = 2$: Let $a \in D(A_{Q(0)}^\alpha)$ with $\alpha > 0$ and $\|f\|_{2,\infty} = \sup_{t>0} \int_t^{t+1} |f(s)|^2_{\mathbb{H}(Q(s))} ds < +\infty$. Then $(\mathrm{Pr.NS})_0$ has a (unique) global strong solution $u(x,t)$ (*2).

(2) The case $n = 3$: Let $a \in D(A_{Q(0)}^\alpha)$ with $\alpha \geq 1/4$ and $\|f\|_{2,\infty} < +\infty$. Then $(\mathrm{Pr.NS})_0$ has a (unique) local strong solution. Moreover, if $|a|_{A_{Q(0)}^\alpha}$ and $\|f\|_{2,\infty}$ are sufficiently small, then the solution can be continued globally. (This result is a natural extension of that of Fujita-Kato [13] for the non-cylindrical case.)

(*2) $u(\cdot,t) \in D(A_{Q(t)})$ for a.e. $t \in R^1$; $\partial u/\partial t$, $\Delta u \in L^2_{loc}((0,+\infty);$ $\mathbb{H}(Q(t)))$; and the zero extension $\hat{u}$ of u to R_x^n satifies $\hat{u} \in C((0,+\infty);\mathbb{H}_\sigma^1(R^n)) \cap C([0,+\infty);\mathbb{H}_\sigma(R^n))$, $\partial\hat{u}/\partial t \in L^2_{loc}((0,+\infty);\mathbb{H}_\sigma(R^n))$.

<u>Sketch of the proof</u>. Let Ω be a bounded auxiliary open ball such that $\overline{Q} \subset \Omega \times R^1$. Let $H = \mathbb{H}_\sigma(\Omega)$ and put

$$\varphi^t(u) = \begin{cases} \frac{1}{2} \sum_{i,j=1}^{n} \int_\Omega \left|\frac{\partial u^i}{\partial x_j}\right|^2 dx & \text{if } u \in \mathbb{H}_\sigma^1(\Omega) \text{ and } u = 0 \text{ a.e. } x \in \Omega \backslash Q(t), \\ +\infty & \text{otherwise}, \end{cases}$$

$$B(t,u) = P_\Omega\,(u \cdot \nabla)\,u \quad \text{with domain } D(B(t,\cdot)) = D(\partial\varphi^t).$$

Then $(\mathrm{Pr.NS})_0$ can be reduced to the following abstract Navier-Stokes problem in $\mathbb{H}_\sigma(\Omega)$:

$$(\mathrm{ANS})_0 \quad \begin{cases} d\hat{u}(t)/dt + \partial\varphi^t(\hat{u}(t)) + B(t,\hat{u}(t)) \ni P_\Omega\,\hat{f}(t), \\ \hat{u}(0) = \hat{a}, \end{cases}$$

where $\hat{f}(\cdot,t)$ and $\hat{a}(\cdot)$ are zero extensions of $f(\cdot,t)$ and $a(\cdot)$ to Ω.

Then (A.1) and (A.2) are easily verified and $(A.\varphi^t)$ with $K = 0$ and $\beta = 1/2$ is assured by (A.Q). Since $(B(t,u), u) = 0$, $(\partial\varphi^t(u), u) = 2\,\varphi^t(u)$ for all $u \in D(\partial\varphi^t)$; and since

$$(2.20) \quad |B(t,u)|_H \le \text{Const.}\, |u|_H^{1/2}\, |\varphi^t(u)|^{1/2} |\partial^\circ\varphi^t(u)|_H^{1/2} \quad \text{for all } u \in D(\partial\varphi^t), \text{ if } n = 2,$$

$$(2.21) \quad |B(t,u)|_H \le \text{Const.}\, |\varphi^t(u)|^{3/4} |\partial^\circ\varphi^t(u)|_H^{1/2} \quad \text{for all } u \in D(\partial\varphi^t), \text{ if } n = 3,$$

(see,e.g., Ladyzhenskaya [19] and Temam [28]) ,

for the case $n = 2$ (resp. $n = 3$) , we can apply Theorem I with $\alpha > 0$ (resp. $\alpha \ge 1/4$) and Theorem IV (resp. V) for $(\mathrm{ANS})_0$. Then the desired solution u is given by $u = \hat{u}|_Q$.

<u>REMARK 2.5</u>. As for the case $n = 4$, it can be also proved that if $|a|_{A^{1/2}_{Q(0)}}$ and $\|f\|_{2,\infty}$ are sufficiently small, then $(\mathrm{Pr.NS})_0$ has a (unique) global strong solution.

§ 3. Periodic Problems.

The same fixed-point method as for $(E)_0$ works well again for this case. Actually, in parallel with Theorems IV and V, we can obtain the following Theorems VI and VII respectively.

THEOREM VI Let (A.1), (A.2), (A.6) and the following $(A.\varphi^t)_\pi$ be satisfied.

$(A.\varphi^t)_\pi$ All conditions in $(A.\varphi^t)$ and the following (i) and (ii) are satisfied.

(i) $\varphi^0(\cdot) = \varphi^T(\cdot)$,

(ii) There exist positive numbers C_1 and p such that $(1<p<+\infty)$

(3.1) $C_1|u|_H^p \leq \varphi^t(u)$ for all $t \in [0,T]$ and $u \in D(\partial\varphi^t)$.

Then, for every $f \in L^2(0,T;H)$, $(E)_\pi$ has a strong periodic solution $u(t)$ satisfying (2.6) and (2.7).

THEOREM VII Let (A.1), (A.2) and $(A.\varphi^t)_\pi$ with $K=0$, $\beta \in [1/2,1]$ and $p \in (1,2]$, and the following (A.8) be satisfied.

(A.8) There exist a function $M(\cdot) \in \mathcal{M}$ and nonnegative numbers α_1, α_2 with $0 \leq \alpha_2 \leq 1$, $2\alpha_1 + \alpha_2 > 1$ such that

(3.2) $|B(t,u)|_H \leq M(|u|_H)|\varphi^t(u)|^{\alpha_1}|\overset{\circ}{\partial}\varphi^t(u)|_H^{\alpha_2}$ for all $t \in [0,T]$ and $u \in D(\partial\varphi^t)$.

Then there exists a (sufficiently small) positive number r such that if $\sup_{1<t<T} \int_{t-1}^{t} |f(s)|_H^2 ds \leq r$, then $(E)_\pi$ has a periodic strong solution $u(t)$ satisfying (2.6) and (2.7).

Application. Let us here consider the periodic problem $(Pr.NS)_\pi$ for the Navier-Stokes equation in $Q_T = \bigcup_{0<t<T} Q(t) \times \{t\}$ with $Q(0) = Q(T)$, i.e., the problem (2.16)-(2.18) with the periodic condition $u(\cdot,0) = u(\cdot,T)$. (This problem is already studied

in Morimoto [22] (in a class of weak solutions) and in [27].)

Results. (1) The case $n=2$: For every $f(t) \in L^2(0,T;\mathbb{H}(Q(t)))$, $(Pr.NS)_\pi$ has a periodic strong solution (by Theorem VI).

(2) The case $n=3$ or 4 : If $\sup_{1<t<T} \int_{t-1}^{t} |f(s)|^2_{\mathbb{H}(Q(s))} ds$ is sufficiently small, then $(Pr.NS)_\pi$ has a (unique) periodic strong solution (by Theorem VII).

Indeed, as for the case $n=4$, we have

$$|B(t,u)|_H \leq \text{Const.} |\varphi^t(u)|^{1/2} |\mathring{\partial}\varphi^t(u)|_H \quad \text{for all } t \in [0,T] \text{ and } u \in D(\partial\varphi^t),$$

which assures (A.8).

§ 4. Almost-Periodic Problems.

Motivation : Let us here reconsider $(Pr.NS)_\pi$. For example, suppose that $\partial Q(t)$, the boundary of $Q(t)$, is composed of two connected hypersurfaces $\Gamma_1(t)$ and $\Gamma_2(t)$ for each t.
When $\partial Q(t)$ moves as t goes on, it would be natural to suppose that the movements of $\Gamma_i(t)$ are independent. Therefore, when the periodic movements of $\Gamma_i(t)$ are discussed, it is rather reasonable to treat the case where the periods ω_i of the movements of $\Gamma_i(t)$ are different. So, if ω_1/ω_2 is not a rational number, then the movement of $\partial Q(t)$ is no longer periodic, but almost-periodic (more pricisely quasi-periodic).
From this point of view, the almost-periodic problem $(E)_{\alpha\pi}$ is regarded as much more important than $(E)_\pi$.

DEFINITION 4.1. (Bohr) A function $v(t) \in C(R^1;H)$ is said to be H-a.p. (H-almost-periodic) if for every $\varepsilon > 0$, there exists a relatively dense set $\{\tau\}_\varepsilon$ in R^1 depending on ε such that

$$\sup_{t \in R^1} |v(t+\tau) - v(t)|_H \leq \varepsilon \quad \text{for all} \quad \tau \in \{\tau\}_\varepsilon$$

Here $\{\tau\}_\varepsilon$ is said to be relatively dense if there exists a positive number ℓ_ε (inclusion length) such that for every $r \in R^1$, the corresponding interval $[r, r+\ell_\varepsilon]$ always contains at least one point of $\{\tau\}_\varepsilon$.

Moreover, a function $w(t) \in L^2_{loc}(R^1;H)$ is said to be $S^2(H)$-a.p. if $\tilde{w}(t) = \{ w(t+\eta) \; ; \; \eta \in [0,1]\}$ is $L^2(0,1;H)$-a.p.

It is well known as Bochner's criterion that the almost-periodicity can be characterized as follows :

THEOREM 4.2. Let $v(t) \in C(R^1;H)$. Then $v(t)$ is H-a.p. if and only if form every sequence $\{\ell_n\}$, there exists a sub-sequence $\{s_n\}$ such that the sequence $\{g(t+s_n)\}$ converges in H uniformly with respect to $t \in R^1$.

Let us here assume that $D(\varphi^t)$ varies almost-periodically in the following sense.

$(A.\varphi^t)_{\alpha\pi}$ For each $t \in R^1$, $\varphi^t \in \Phi(H)$ and $\varphi^t \geq 0$. Furthermore there exist R^1-almost-periodic functions $h_1(\cdot), h_2(\cdot) \in W^{1,\infty}(R^1)$ and a continuous function $m(\cdot) \in \mathcal{M}$ such that for every $t_0 \in R^1$ $x_0 \in D(\varphi^{t_0})$, there eists a function $x(t)$ on R^1 such that

$$(4.1) \quad |x(t) - x_0|_H \leq m(|x_0|_H)\,|h_1(t) - h_1(t_0)|\,(\varphi^{t_0}(x_0) + 1) ,$$

$$(4.2) \quad \varphi^t(x(t)) \leq \varphi^{t_0}(x_0) + m(|x_0|_H)\,|h_2(t) - h_2(t_0)|\,(\varphi^{t_0}(x_0) + 1) ,$$

for all $t \in R^1$.

In addition, we assume .

(A.9) The following (i)-(iii) are satisfied.

(i) $\varphi^t(0) = 0$ for all $t \in R^1$,

(ii) $C_1 |u|_H^p \leq \varphi^t(u)$, $C_1 > 0$, $1 < p < +\infty$, for all $u \in D(\partial\varphi^t)$,

(iii) $(g_1 - g_2, u_1 - u_2)_H \geq C |u_1 - u_2|_H^2$, $C > 0$, for all $t \in R^1$ $u_i \in D(\partial\varphi^t)$ and $g_i \in \partial\varphi^t(u_i)$

Then , concerning the unperturbed problem $(E)_{\alpha\pi}$ with $B(t,\cdot) \equiv 0$, we have :

<u>THEOREM VIII</u> Let $(A.\varphi^t)_{\alpha\pi}$,(A.1) and (A.9) be satisfied. Let $f(t)$ be[(*3)] $S^2(H)$-a.p. Then $(E)_{\alpha\pi}$ with $B(t,\cdot) \equiv 0$ has a unique H-almost-periodic strong solution.

Since the unperturbed problem is solved as above, in order to solve $(E)_{\alpha\pi}$, we intend to apply the same fixed-point method as in § 2. Unfortunately, however, in this procedure some difficulties arise. For example, it is difficult to know if $\mathbb{B}(h)(t)$ is almost-periodic (in some sense) when $h(t)$ is almost-periodic , and how to take a (weakly) compact set such as $K^{\alpha}_{S,R}$ where $\mathbb{B}$ works. Therefore we here apply another method similar to that in Biroli [6] : Firstly, the existence and (local) uniqueness of bounded solutions are shown. Next, the unique bounded solution is proved to be almost-periodic by using Bochner's criterion. Nevertheless this method requires so restrictive conditions on $\partial\varphi^t$ and $B(t,\cdot)$ that we give up to present our results in abstract forms. So we here only illustrate this method for the Navier-Stokes problem (Pr.NS) in regions with almost-periodically moving boundaries.

(*3) This can be replaced by $S^2(H)$-a.p. in a weak topology.

$$
(\mathrm{Pr.NS})\begin{cases}(4.3) & \dfrac{\partial u}{\partial t} - \Delta u + (u\cdot\nabla)u = f - \nabla p_* & \text{in } Q,\\ (4.4) & \operatorname{div} u = 0 & \text{in } Q,\\ (4.5) & u = 0 & \text{on } \Gamma = \bigcup_{t\in R^1} Q(t)\times\{t\},\end{cases}
$$

where $Q(t)$ moves almost-periodically in the following sense.

$(A.Q)_{a\pi}$ All conditions (i)-(iii) of (A.Q) and the following (iv) be satisfied.

(iv) $D^m F(x,t)$ $(m=0,1,2,3)$ are almost-periodic in t uniformly with respect to $x \in \overline{Q(0)}$, i.e., for every $\varepsilon > 0$, there exists a relatively dense set $\{\tau\}_\varepsilon$ such that

$$|D^m F(x,t+\tau) - D^m F(x,t)| < \varepsilon \quad \text{for all } \tau \in \{\tau\}_\varepsilon \text{ and all } (x,t)\in \overline{Q(0)}\times R^1.$$

Then our result is stated as follows.

THEOREM 4.3. Let $n = 2, 3$ or 4 and $(A.Q)_{a\pi}$ be satisfied. Then there exists a (sufficiently small) positive number r such that if $\|f\|_{2,\infty} = \sup_{t\in R^1} \int_{t-1}^{t} |f(s)|^2_{\mathbb{H}(Q(s))} ds \le r$ and $\hat{f}(t)$ is $S^2(\mathbb{H}(\Omega))$-a.p. (*3), then (Pr.NS) has a (unique) strong solution $u(t)$ such that the zero extension $\hat{u}(t)$ of $u(t)$ is $\mathbb{H}_\sigma(\Omega)$-a.p.

Sketch of proof. If $\|f\|_{2,\infty} = r$ is sufficiently small, then Theorem V assures that there exist strong solutions $\hat{u}_n(t)$ in $(-n,+\infty)$ of the abstract Navier-Stokes problems in $H = \mathbb{H}_\sigma(\Omega)$:

$$
\begin{cases} d\hat{u}_n(t)/dt + \partial\varphi^t(\hat{u}_n(t)) + B(t,\hat{u}_n(t)) \ni P_\Omega \hat{f}(t) & t \in (-n,+\infty),\\ \hat{u}_n(-n) = 0. \end{cases}
$$

Then, by Lemma 2.4, as a limit of $\hat{u}_n(t)$ we can construct a bounded strong solution $\hat{u}(t)$ in R^1 of the abstract Navier-Stokes problem such that

(4.6) $\sup_{t\in R^1} \int_{t-1}^{t} |\frac{d\hat{u}}{ds}(s)|_H^2 \, ds < +\infty$,

(4.7) $\sup_{t\in R^1} (|\hat{u}(t)|_H + \varphi^t(\hat{u}(t))) \leq N r$.

Now we are going to show that this bounded solution $\hat{u}(t)$ is H-a.p. Suppose that $\hat{u}(t)$ is not H-a.p. , then by Bochner's criterion and the almost-periodicity of $\hat{f}(t)$ and $F(\cdot,t)$, there exist sequences $\{\ell_j\}$, $\{t_j\}$ and subsequences $\{\ell_{ij}\}$ of $\{\ell_j\}$ ($i=1,2$) such that

(4.8) $|\hat{u}(t_j+\ell_{1j}) - \hat{u}(t_j+\ell_{2j})|_H \geq \rho > 0$ for all j ,

(4.9) $P_\Omega \hat{f}(t+\tau_{ij}) \to \hat{f}_\ell(t)$ in $L^2_{loc}(R^1;H)$ uniformly in $t\in R^1$ as $j \to +\infty$,

(4.10) $D^m F^k(x,t+\tau_{ij}) \to D^m F^k_\ell(x,t)$ uniformly in $(x,t)\in Q(0)xR^1$ as $j \to +\infty$, for $m=0,1,2,3$ $k=1,2,\cdot\cdot,n$.

where we put $\tau_{ij} = t_j + \ell_{ij}$.

Put $Q_\ell(t) = \bigcup_{x\in Q(0)} F_\ell(x,t)$ and $u_{ij}(t) = \hat{u}(t+\tau_{ij})$. Then $Q_\ell(t)$ forms another smooth non-cylindrical domain. Moreover, from (4.6),(4.7) and (A.1), there exist subsequences $\{u_{ij'}(t)\}$ of $\{u_{ij}(t)\}$ such that $u_{ij'}(t)$ converge to $u_i(t)$ which satisfy

(4.11) $(\frac{du_i}{dt}(t), \phi(t)) + (\nabla u_i(t), \nabla\phi(t)) - ((u_i(t)\cdot\nabla) \phi(t), u_i(t))$

$= (\hat{f}_\ell(t), \phi(t))$ for a.e. $t \in R^1$ and all $\phi(t) \in \mathbb{H}^1_\sigma(Q_\ell(t))$.

Then, putting $\phi(t) = w(t) = u_1(t) - u_2(t)$ in (4.11), we have

(4.12) $\frac{1}{2}\frac{d}{dt} |w(t)|^2 + |\nabla w(t)|^2 \leq - ((w(t)\cdot\nabla)u_1(t), w(t))$

$\leq$ Const. $|\nabla w(t)|^2 |\nabla u_1(t)|$.

That is to say, by (4.7), for a sufficiently small r ,

$$\frac{d}{dt} |w(t)|^2 + |\nabla w(t)|^2 \leq 0 .$$

Hence $|w(t)|$ is monotone decreasing and

$$\int_{t_1}^{t_2} |\nabla w(t)|^2 \, dt \leq |w(t_1)|^2 - |w(t_2)|^2 \quad \text{for all } t_1 \text{ and } t_2 .$$

Since $|w(t_i)|$ are bounded, letting $t_1 \to -\infty$, we find that $|\nabla w(t)| \to 0$, i.e., $|w(t)| \to 0$ as $t \to -\infty$. Thus we have $|w(0)| \leq \lim_{t \to -\infty} |w(t)| = 0$, which contradicts (4.8).

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